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Results for intrabeam scattering growth rates for a bi-gaussian distribution

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Abstract

This note lists results for the intrabeam scattering growth rates for a bi-gaussian distribution. The derivation of these results will be given in a future note.

Introduction

This note lists results for the intrabeam scattering growth rates for a bi-gaussian distribution. The derivation of these results will be given in a future note.

The bi-gaussian distribution is interesting for studying the possibility of using electron cooling in RHIC. Studies done using the SIMCOOL program [1] indicate that in the presence of electron cooling, the beam distribution changes so that it developes a strong core and a long tail which is not described well by a gaussian, but may be better described by a bi-gaussian. Being able to compute the effects of intrabeam scattering for a bi-gaussian distribution would be useful in computing the effects of electron cooling, which depend critically on the details of the intrabeam scattering.

Gaussian distribution

Before defining the bi-gaussian distribution, the gaussian distribution will be reviewed.

$Nf(x,p)$ gives the number of particles in d^3xd^3p, where N is the number of particles in a bunch. For a gaussian distribution, $f(x,p)$ ls given by

$$f(x,p) = \frac{1}{\Gamma} exp[-S(x,p)] \tag{1}$$

$$\begin{aligned}
S &= S_x + S_y + S_s \\
S_x &= \frac{1}{\bar{\epsilon_x}} \epsilon_x(x_\beta, p_{x\beta}/p_0) \\
x_\beta &= x - D(p-p_0)/p_0 \\
p_{x\beta}/p_0 &= p_x/p_0 - D'(p-p_0)/p_0 \\
\epsilon_x(x,x') &= \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 \\
S_y &= \frac{1}{\bar{\epsilon_y}} \epsilon_y(y, p_y/p_0) \\
\epsilon_y(y,y') &= \gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2 \\
S_s &= \frac{1}{2\sigma_s^2}(s-s_c)^2 + \frac{1}{2\sigma_p^2}((p-p_0)/p_0)^2 \\
S_s &= \frac{1}{\bar{\epsilon_s}}(\frac{1}{\beta_s}(s-s_c)^2 + \beta_s((p-p_0)/p_0)^2 \\
\beta_s &= \sigma_s/\sigma_p \\
\bar{\epsilon_s} &= 2\sigma_s\sigma_p \\
S_s &= \frac{1}{\bar{\epsilon_s}} \epsilon_s(s-s_c, (p-p_0)/p_0) \\
\Gamma &= \int d^3xd^3p \; exp[-S(x,p)] \\
\Gamma &= \pi^3 \bar{\epsilon_x}\bar{\epsilon_y}\bar{\epsilon_s} p_0^3 \\
\bar{\epsilon_i} &= <\epsilon_i(x,p)> \quad i = x,y,s
\end{aligned} \tag{2}$$

D is the horizontal dispersion. $D' = dD/ds$. $<>$ indicates an average over all the particles in a bunch.

Growth rates for a Gaussian distribution

In the following,the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for $< p_i p_j >$. From these one can compute the growth rates for $< \epsilon_i >$ using the relations given at the end of this note.

$$
\begin{aligned}
\frac{1}{p_0^2}\frac{d}{dt} < p_i p_j > &= \frac{N}{\Gamma}\int d^3\Delta \ exp[-R]C_{ij} \\
C_{ij} &= \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2(|\Delta|^2\delta_{ij} - 3\Delta_i\Delta_j)2\bar{\beta}c \ \ ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2] \\
\bar{\beta} &= \beta_0\gamma_0|\Delta/p_0| \\
r_0 &= Z^2e^2/Mc^2 \\
R &= R_x + R_y + R_s \\
R_x &= \frac{2}{\beta_x\bar{\epsilon_x}}[\gamma^2 D^2\Delta_s^2 + (\beta_x\Delta_x - \gamma\tilde{D}\Delta_s)^2]/p_0^2 \\
\tilde{D} &= \beta_x D' + \alpha_x D \\
R_y &= \frac{2}{\beta_y\bar{\epsilon_y}}\beta_y^2\Delta_y^2/p_0^2 \\
R_s &= \frac{2}{\beta_s\bar{\epsilon_s}}\beta_s^2\gamma^2\Delta_s^2/p_0^2
\end{aligned}
\qquad (3)
$$

The integral over $d^3\Delta$ is an integral over all possible values of the relative momemtum for any two particles in a bunch.β_0, γ_0 are the beta and gamma corresponding to p_0, the central momemtum of the bunch in the Laboratory Coordinate System. $\gamma = \gamma_0$

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$ and using $d^3\Delta = |\Delta|^2 d|\Delta| sin\theta d\theta d\phi$.

$$
\begin{aligned}
\frac{1}{p_0^2}\frac{d}{dt} < p_i p_j > &= \frac{N}{\Gamma}2\pi p_0^3\left(\frac{r_0}{2\gamma_0^2\beta_0^2}\right)^2 2\beta_0\gamma_0 c\int sin\theta d\theta d\phi \ (\delta_{ij} - 3g_i g_j) \\
&\quad \frac{1}{F}ln\left[\frac{\hat{C}}{F}\right] \\
g_3 &= cos\theta = g_s \\
g_1 &= sin\theta cos\phi = g_x \\
g_2 &= sin\theta sin\phi = g_y \\
\hat{C} &= 2\gamma_0^2\beta_0^2 b_{max}/r_0 \\
F &= R/(|\Delta|/p_0)^2
\end{aligned}
$$

$$
\begin{aligned}
F &= F_x + F_y + F_s \\
F_x &= \frac{2}{\beta_x \bar{\epsilon_x}}[\gamma^2 D^2 g_s^2 + (\beta_x g_x - \gamma \tilde{D} g_s)^2] \\
F_y &= \frac{2}{\beta_y \bar{\epsilon_y}} \beta_y^2 g_y^2 \\
F_s &= \frac{2}{\beta_s \bar{\epsilon_s}} \beta_s^2 \gamma^2 g_s^2
\end{aligned} \tag{4}
$$

Bi-Gaussian distribution

The bi-gaussian distribution will be assumed to have the form given by the following.

$Nf(x,p)$ gives the number of particles in d^3xd^3p, where N is the number of particles in a bunch. For a bi-gaussian distribution, $f(x,p)$ ls given by

$$
f(x,p) = \frac{N_a}{N}\frac{1}{\Gamma_a} exp[-S_a(x,p)] + \frac{N_b}{N}\frac{1}{\Gamma_b} exp[-S_b(x,p) \tag{5}
$$

In the first gaussian,to find Γ_a, S_a then in the expressions for Γ, S, given above for the gaussian distribution, replace $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$ by $\bar{\epsilon_{xa}}, \bar{\epsilon_{ya}}, \bar{\epsilon_{sa}}$. In the second gaussian, in the expressions for Γ, S, replace $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$ by $\bar{\epsilon_{xb}}, \bar{\epsilon_{yb}}, \bar{\epsilon_{sb}}$. In addition. $N_a + N_b = N$. This bi-gaussian has 7 parameters instead of the three parameters of a gaussian.

Growth rates for a Bi- Gaussian distribution

In the following,the growth rates are given in the Rest Coordinate System, which is the coordinate system moving along with the bunch. Growth rates are given for $< p_i p_j >$. From these one can compute the growth rates for $< \epsilon_i >$ using the relations given at the end of this note.

$$
\begin{aligned}
\frac{1}{p_0^2}\frac{d}{dt} < p_i p_j > &= N \int d^3\Delta \; C_{ij} [\left(\frac{N_a}{N}\right)^2 \frac{exp(-R_a)}{\Gamma_a} + \left(\frac{N_b}{N}\right)^2 \frac{exp(-R_b)}{\Gamma_b} \\
&\quad + 2 \frac{N_a N_b}{N^2} \frac{\Gamma_c}{\Gamma_a \Gamma_b} exp(-T)] \\
\frac{1}{\bar{\epsilon_{ic}}} &= \frac{1}{2}\left(\frac{1}{\bar{\epsilon_{ia}}} + \frac{1}{\bar{\epsilon_{ib}}}\right) \quad i = x, y, s
\end{aligned}
$$

$$
\begin{aligned}
T &= T_x + T_y + T_s \\
T_x &= R_{xc}\bar{\epsilon_a}\bar{\epsilon_b}/\bar{\epsilon_c}^2 \\
T_y &= R_{yc} \\
T_s &= R_{sc} \\
C_{ij} &= \frac{2\pi}{p_0^2}(r_0/2\bar{\beta}^2)^2(|\Delta|^2\delta_{ij} - 3\Delta_i\Delta_j)2\bar{\beta}c \;\; ln[1 + (2\bar{\beta}^2 b_{max}/r_0)^2] \\
\bar{\beta} &= \beta_0\gamma_0|\Delta/p_0| \\
r_0 &= Z^2e^2/Mc^2
\end{aligned}
\tag{6}
$$

R_a, R_b, R_c are each the same R that was defined for the Gaussian distribution except that $\bar{\epsilon_i}$ are replaced by $\bar{\epsilon_{ia}}, \bar{\epsilon_{ib}}, \bar{\epsilon_{ic}}$ respectively.

The above 3-dimensional integral can be reduced to a 2-dimensional integral by integrating over $|\Delta|$.

$$
\begin{aligned}
\frac{1}{p_0^2}\frac{d}{dt} < p_i p_j > &= 2\pi p_0^3 \left(\frac{r_0}{2\gamma_0^2\beta_0^2}\right)^2 2\beta_0\gamma_0 c \int sin\theta d\theta d\phi \; (\delta_{ij} - 3g_i g_j) \\
& N[\left(\frac{N_a}{N}\right)^2 \frac{1}{\Gamma_a F_a} ln[\frac{\hat{C}}{F_a}] + \left(\frac{N_b}{N}\right)^2 \frac{1}{\Gamma_b F_b} ln[\frac{\hat{C}}{F_b}] \\
& +2\frac{N_a N_b}{N^2}\frac{\Gamma_c}{\Gamma_a\Gamma_b}\frac{1}{G} ln[\frac{\hat{C}}{G}]] \\
g_3 &= cos\theta = g_s \\
g_1 &= sin\theta cos\phi = g_x \\
g_2 &= sin\theta sin\phi = g_y \\
\hat{C} &= 2\gamma_0^2\beta_0^2 b_{max}/r_0 \\
G &= T/(|\Delta|/p_0)^2 \\
G &= F_{xc}(\bar{\epsilon_{xa}}\bar{\epsilon_{xb}}/\bar{\epsilon_{xc}}^2) + F_{yc} + F_{sc}
\end{aligned}
\tag{7}
$$

F_a, F_b, F_c are each the same F that was defined for the Gaussian distribution except that the $\bar{\epsilon_i}$ are replaced by $\bar{\epsilon_{ia}}, \bar{\epsilon_{ib}}, \bar{\epsilon_{ic}}$ respectively.

The above results for the growth rates for a bi-gaussian distribution are expressed as an integral which contains 3 terms, each of which is similar to the one term in the results for the gaussian distribution. These three terms may be given a simple interpertation. The first term represents the contribution to the growth rates due to the scattering of the N_a particles of the first gaussian from themselves, the seond term the contribution due to the scattering of the N_b particles of the second gaussian from themselves, and the third term the contribution due to the scattering of the N_a particles of the first gaussian from the N_b partcles of the second gaussian.

Emittance growth rates

One can compute growth rates for the average emittances, $<\epsilon_i>$ in the Laboratory Coordinate System, from the growth rates for $<p_i p_j>$ in the Rest Coordinate System.In the following , dt is the time interval in the Laboratory System and $d\tilde{t}$ is the time interval in the Rest System. $dt = \gamma d\tilde{t}$

$$
\begin{aligned}
\frac{d}{dt}\epsilon_x &= \frac{\beta_x}{\gamma}\frac{d}{d\tilde{t}} < p_x^2/p_0^2 > + \frac{D^2+\tilde{D}^2}{\beta_x}\gamma\frac{d}{d\tilde{t}} < p_s^2/p_0^2 > -2\tilde{D}\frac{d}{d\tilde{t}} < p_x p_s/p_0^2 > \\
\frac{d}{dt}\epsilon_y &= \frac{\beta_y}{\gamma}\frac{d}{d\tilde{t}} < p_y^2/p_0^2 > \\
\frac{d}{dt}\epsilon_s &= \frac{\beta_s}{\gamma}\frac{d}{d\tilde{t}} < p_s^2/p_0^2 >
\end{aligned} \tag{8}
$$

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References

1. V.Parhomchhuk and I. Ben-Zvi, BNL report C-A/AP/47, April 2001; A. Fedotov, Y. Eidelman (Private Communication 2004)